THE TRACE FORMULA OF PETERSSON

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1. Introduction

The Petersson trace formula provides a deep relationship between the Fourier coefficients of cusp forms on one hand, and Bessel functions and Kloosterman sums on the other hand. **In this expository note, we'll prove this formula.** Following Iwaniec and Kowalski's exposition in [3], we'll do this by computing the Fourier coefficients of the Poincaré series

$$P_m(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(q)} \overline{\chi}(\gamma) j_\gamma(z)^{-k} e(m\gamma z)$$

in two different ways:

(1) On one hand, we'll express the n'th Fourier coefficient of $P_m(z)$ as an infinite series of Bessel functions twisted by scaled Kloosterman sums. This is carried out in Section 2, where we construct Poincaré series, and in Section 3, where we decompose the coset space $\Gamma_\infty \backslash \Gamma_0(q)$ in order to more easily get at the n'th Fourier coefficient. Appendix A verifies a contour shift that furnishes the integral representation for the J_ν-Bessel function that is used in this section.

(2) On the other hand, we'll spectrally expand $P_m(z)$ inside the Hilbert space of cusp forms, and then compute that the n'th Fourier coefficient of this spectral expansion is a scaled "inner product" of Fourier coefficients of any basis of the Hilbert space. These details are carried out in Section 4, where we construct the Petersson inner product on the space of cusp forms, and in Section 5, where we compute the projection of an arbitrary cusp form onto a Poincaré series.

We combine these calculations in Section 6 to prove Petersson's trace formula. And in Section 7, we provide a toy application of Petersson's trace formula: a convergent series for the Ramanujan tau function. Throughout this note, we freely use basic definitions and notations from the theory of modular forms, for which a good reference is [2].

2. Construction of Poincaré series

Here we define the Poincaré series and gather some of its basic properties. Let us fix integers $m \geq 0$ and $k > 2$. If χ is a Dirichlet character modulo q (not necessarily primitive) then clearly χ induces a character of $\Gamma_0(q)$ by $\chi(\gamma) := \chi(d)$, where here and throughout, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Define the automorphy factor $j_\gamma(z) := cz + d$, as well as the matrix group $\Gamma_\infty := \{\pm T^n : n \in \mathbb{Z}\}$, where $T_n := \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. We write $e(x) := e^{2\pi i x}$.

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Proposition 2.1. *The m'th Poincaré series (of weight k, level q, twisted by character χ), which is defined as*

$$P_m(z) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(q)} \overline{\chi}(\gamma) j_\gamma(z)^{-k} e(m\gamma z), \tag{2.1}$$

has the following properties:

1. *The summands are well-defined (independent of choice of coset representative);*
2. *$P_m(z)$ converges absolutely in $\mathbb{H}$ by comparison to the Eisenstein series $E_k(z)$;*
3. *For any $\tau \in \Gamma_0(q)$, we have $P_m(\tau z) = \chi(\tau) j_\tau(z)^k P_m(z)$.*

Proof. We first verify that the summands of P_m are well-defined. If $\Gamma_\infty\gamma_1 = \Gamma_\infty\gamma_2$, we must show that

$$\overline{\chi}(d_1)(c_1 z + d_1)^{-k} e(m\gamma_1 z) = \overline{\chi}(d_2)(c_2 z + d_2)^{-k} e(m\gamma_2 z). \tag{2.2}$$

Towards this, we'll use the fact that the following is a well-defined bijection:

$$\begin{aligned} \Gamma_\infty \backslash \Gamma_0(q) &\to \left\{(c,d) \in \mathbb{Z}^2 : c > 0, q \mid c, (c,d) = 1\right\} \cup \{(0,1)\} \\ \Gamma_\infty \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto \begin{cases} (0,1) & c = 0 \\ (|c|, \operatorname{sgn}(c) \cdot d) & c \neq 0. \end{cases} \end{aligned} \tag{2.3}$$

The proof of this is elementary, and is worked out in full detail for $q = 1$ in [1]; the general case follows *mutatis mutandis.*

If $c_1 = c_2$ and $d_1 = d_2$ in (2.2), then the first two factors cancel; if $c_1 = -c_2$ and $d_1 = -d_2$, then the first two factors cancel iff $\chi(-1) = (-1)^k$, which we assume.[1] Therefore it suffices to show that $e(m\gamma z) = e(m(\pm T^n \gamma z))$. We compute that

$$T^n \gamma z = \begin{pmatrix} a + nc & b + nd \\ c & d \end{pmatrix} z = \frac{(a+nc)z + (b+nd)}{cz+d} = \frac{az+b}{cz+d} + \frac{n(cz+d)}{cz+d} = \gamma z + n,$$

which implies that $e(m\gamma z) = e(mT^n\gamma z)$ by periodicity of the exponential map. And as $-I$ acts trivially on $\mathbb{H}$, we have $(-T^n\gamma)z = (T^n\gamma)z$, so $e(m\gamma z) = e(m(-T^n)\gamma z)$ as well.

The Poincaré series converges absolutely, as can be seen by comparison to the Eisenstein series: for any $\gamma \in \Gamma_\infty\backslash\Gamma_0(q)$, we have $\left|e^{2\pi i m(\gamma z)}\right| \leq 1$ because $\gamma z = x + iy \in \mathbb{H}$ and $\left|e^{2\pi i m(x+iy)}\right| = e^{-2\pi m y} < 1$. And the modular transformation law is a straightforward consequence of absolute convergence: we compute

$$\begin{aligned} P_m(\tau z) &= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(q)} \overline{\chi}(\gamma) j_\gamma(\tau z)^{-k} e(m\gamma\tau z) \\ &= \chi(\tau) j_\tau(z)^k \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(q)} \overline{\chi}(\gamma\tau) j_{\gamma\tau}(z)^{-k} e(m\gamma\tau z), \end{aligned}$$

which follows from the chain rule $j_{\gamma\tau}(z) = j_\gamma(\tau z) j_\tau(z)$. By absolute convergence, we can rearrange this series to obtain the transformation law (3), as required. ∎

[1] Standard fact: by the modular transformation law with $-I$, modular forms vanish everywhere unless $\chi(-1) = (-1)^k$.

3. Fourier expansion of Poincaré series

Expanding the coset sum defining $P_m(z)$ into an infinite series will require decomposing the Fuschian group $\Gamma_0(q)$ in various ways. This is taken care of in the following group-theoretic lemma.

Lemma 3.1. *The group $\Gamma_0(q)$ decomposes as the disjoint union*

$$\Gamma_0(q) = \Gamma_\infty \cup \bigcup_{\substack{c>0 \\ q|c}} \bigcup_{\substack{1\le d\le c \\ (c,d)=1}} \Gamma_\infty \begin{pmatrix} * & * \\ c & d \end{pmatrix} \Gamma_\infty. \tag{3.1}$$

In particular, this implies the following decompositions:

$$\Gamma_\infty\backslash\Gamma_0(q)/\Gamma_\infty = \{\Gamma_\infty\} \cup \left\{\Gamma_\infty \begin{pmatrix} * & * \\ c & d \end{pmatrix} \Gamma_\infty : c > 0, 1 \le d \le c \text{ with } q \mid c, (c,d) = 1\right\}$$

$$\Gamma_\infty\backslash\Gamma_0(q) = \{\Gamma_\infty\} \cup \left\{\Gamma_\infty \begin{pmatrix} * & * \\ c & d \end{pmatrix} : c > 0, d \in \mathbb{Z} \text{ with } q \mid c, (c,d) = 1\right\}.$$

Proof. We'll first argue that set equality holds in (3.1). Consider

$$\gamma = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Gamma_0(q), \tag{3.2}$$

and assume $\gamma \notin \Gamma_\infty$, so $c' \neq 0$. If $c' > 0$, then by the division algorithm, let us write $d' = nc' + d$ for some $1 \le d \le c'$, so

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a' & b'-na' \\ c' & d'-nc' \end{pmatrix} = \begin{pmatrix} a' & b'-na' \\ c' & d \end{pmatrix},$$

hence

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a' & b'-na' \\ c' & d \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \in \Gamma_\infty \begin{pmatrix} a' & b'-na' \\ c' & d \end{pmatrix} \Gamma_\infty.$$

And if $c' < 0$, then writing $-d' = n(-c') + d$, where $1 \le d \le -c'$, we compute

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -a' & -b'+na' \\ -c' & -d'+nc' \end{pmatrix} = \begin{pmatrix} -a' & -b'+na' \\ -c' & d \end{pmatrix},$$

hence

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -a' & -b'+na' \\ -c' & d \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \in \Gamma_\infty \begin{pmatrix} -a' & -b'+na' \\ -c' & d \end{pmatrix} \Gamma_\infty.$$

This shows that the RHS of (2.3) contains $\Gamma_0(q)$.

Towards verifying the converse, let us consider the following cosets:

$$\Gamma_\infty \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left\{\pm \begin{pmatrix} a+nc & b+nd \\ c & d \end{pmatrix}\right\}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\Gamma_\infty = \left\{\pm \begin{pmatrix} a & b+na \\ c & d+nc \end{pmatrix}\right\}. \tag{3.3}$$

Together, these computations imply that every element of the double coset

$$\Gamma_\infty \begin{pmatrix} * & * \\ c & d \end{pmatrix} \Gamma_\infty$$

has lower left entry $\pm c$; in particular, that $q \mid c$ in the RHS of (3.1) implies $\Gamma_\infty\gamma\Gamma_\infty \subseteq \Gamma_0(q)$, so equality indeed holds in (3.1).

That (3.1) is indeed a disjoint union follows from the fact that two double cosets $\Gamma_\infty\gamma\Gamma_\infty$ and $\Gamma_\infty\gamma'\Gamma_\infty$ are either disjoint or identical. Namely, by (3.3), for a given coset $\Gamma_\infty\gamma'\Gamma_\infty$, every lower left matrix entry is equal to $|c'|$, and every lower right matrix extry is equivalent to d' (mod $|c'|$). ☐

One can prove [5] that $P_m(z)$ vanishes at all cusps not equivalent to ∞. For $m \geq 1$, the following result shows that it vanishes at the cusp ∞ as well. The statement of this result is adapted[2] from [3, Lemma 11.2].

Proposition 3.2. *For $m \geq 1$, the Poincaré series $P_m(z)$ has the Fourier expansion*

$$P_m(z) = \sum_{n\geq 1} p_m(n)e(nz),$$

with coefficients given by

$$p_m(n) = \delta(m,n) + \left(\frac{n}{m}\right)^{\frac{k-1}{2}} 2\pi \cdot i^k \sum_{\substack{c>0 \\ q|c}} c^{-1} S_{\overline{\chi}}(m,n;c) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right), \tag{3.4}$$

where $\delta(m,n)$ is the Kronecker delta function, S_χ is the Kloosterman sum defined in (3.7), and J_{k-1} is the Bessel function defined in (3.13).

Proof. Our starting point is the expansion

$$P_m(z) = e(mz) + \sum_{1\neq\gamma\in\Gamma_\infty\backslash\Gamma_0(q)} \overline{\chi}(\gamma) j_\gamma(z)^{-k} e(m\gamma z). \tag{3.5}$$

Define $\Gamma'_\infty := \{T^n : n \in \mathbb{Z}\}$ to be the positive half of Γ_∞. We will decompose the above coset sum using a 1:1 correspondence between single and double coset representatives

$$\{\gamma : \Gamma_\infty \neq \Gamma_\infty\gamma \in \Gamma_\infty\backslash\Gamma_0(q)\} = \{\gamma'\tau : \Gamma_\infty \neq \Gamma_\infty\gamma'\Gamma_\infty \in \Gamma_\infty\backslash\Gamma_0(q)/\Gamma_\infty, \tau \in \Gamma'_\infty\} \tag{3.6}$$

provided by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b' \\ c & d' \end{pmatrix} \cdot \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b' + an \\ c & d' + cn \end{pmatrix};$$

here, we may assume $c > 0$ because of the left action of $-I \in \Gamma_\infty$, as well as because both γ and $-\gamma$ yield identical summands in (3.5); n and d' are chosen so that $d = cn + d'$ with $1 \leq d' \leq c$; and b' is chosen to satisfy $ad' - b'c = 1$. This map (3.6) is a bijection because of the double coset decomposition (3.1). Note that this correspondence factors a given coset representative; in particular, the image of a coset representative is equal to the input. Using this correspondence, we can decompose the Poincaré series (3.5) as follows:

$$\begin{aligned} P_m(z) &= e(mz) + \sum_{1\neq\gamma\in\Gamma_\infty\backslash\Gamma_0(q)/\Gamma_\infty} \sum_{\tau\in\Gamma'_\infty} \overline{\chi}(\gamma\tau) j_{\gamma\tau}(z)^{-k} e(m\gamma\tau z) \\ &= e(mz) + \sum_{1\neq\gamma\in\Gamma_\infty\backslash\Gamma_0(q)/\Gamma_\infty} \sum_{n\in\mathbb{Z}} \overline{\chi}(\gamma) j_\gamma(z+n)^{-k} e(m\gamma(z+n)), \end{aligned}$$

[2]Proposition 3.2 corrects a typo in the Kloosterman sum from the corresponding result in [3].

because for all $\tau \in \Gamma'_\infty$, we have $\overline{\chi}(\tau) = 1$ and $j_{\gamma\tau}(z) = j_\gamma(\tau z) j_\tau(z) = j_\gamma(\tau z)$. Next we use Lemma 3.1 to transform this double coset sum into an infinite series, which yields

$$\begin{aligned} P_m(z) &= e(mz) + \sum_{\substack{c>0 \\ q|c}} \sum_{\substack{1\le d\le c \\ (c,d)=1}} \overline{\chi}(d) \sum_{n\in\mathbb{Z}} (c(z+n)+d)^{-k} e\left(m\left(\frac{a(z+n)+b}{c(z+n)+d}\right)\right) \\ &= e(mz) + \sum_{\substack{c>0 \\ q|c}} \sum_{\substack{1\le d\le c \\ (c,d)=1}} \overline{\chi}(d) \sum_{n\in\mathbb{Z}} (c(z+n)+d)^{-k} e\left(m\left(\frac{a}{c} - \frac{1}{c(c(z+n)+d)}\right)\right). \end{aligned}$$

By our computation (3.3), we know a is uniquely determined modulo c due to the left action of Γ_∞, so these summands are well-defined by periodicity of the exponential map.

Let us denote by $I_n(c,d;z)$ the innermost summand. We compute the Fourier transform $\hat{I}_n(c,d;z) = \int_{\mathbb{R}} I_v(c,d;z) e^{-nv} dv$ to be

$$\begin{aligned} \hat{I}_n(c,d;z) &= \int_{\mathbb{R}} (c(z+v)+d)^{-k} e\left(\frac{am}{c} - \frac{m}{c(c(z+v)+d)} - nv\right) dv \\ &= e\left(\frac{am}{c} + \frac{nd}{c}\right) \left(\int_{-\infty+iy}^{\infty+iy} (cu)^{-k} e\left(\frac{-m}{c^2u} - nu\right) du\right) e(nz) \end{aligned}$$

via the change of variables $cu = c(z+v)+d$. Applying Poisson summation, we can continue:

$$P_m(z) = e(mz) \sum_{\substack{c>0 \\ q|c}} \sum_{\substack{1\le d\le c \\ (c,d)=1}} \overline{\chi}(d) \sum_{n\in\mathbb{Z}} e\left(\frac{am}{c} + \frac{nd}{c}\right) \left(\int_{-\infty+iy}^{\infty+iy} (cu)^{-k} e\left(\frac{-m}{c^2u} - nu\right) du\right) e(nz).$$

If we define the Kloosterman sum

$$S_\chi(m,n;c) := \sum_{\substack{1\le d\le c \\ (c,d)=1}} \chi(d) e\left(\frac{d^{-1}m}{c} + \frac{dn}{c}\right), \tag{3.7}$$

where d^{-1} is taken modulo c, then we can express this as

$$P_m(z) = e(mz) + \sum_{n\in\mathbb{Z}} \sum_{\substack{c>0 \\ q|c}} S_{\overline{\chi}}(m,n;c) \left(\int_{-\infty+iy}^{\infty+iy} (cu)^{-k} e\left(\frac{-m}{c^2u} - nu\right) du\right) e(nz) \tag{3.8}$$

because the integral is independent of d.

We'll now argue that the integral in (3.8) vanishes for $n \le 0$, by computing the following limit:

$$\lim_{t\to\infty} \int_{-t+iy}^{t+iy} (cu)^{-k} e\left(\frac{-m}{c^2u} - nu\right) du = 0. \tag{3.9}$$

We'll accomplish this by shifting the contour upwards. Towards this, fix $t > y$ and consider the rectangle in $\mathbb{H}$ with vertices

$$-t+iy, \qquad -t+it, \qquad t+it, \qquad t+iy.$$

In the limit, the vanishing of the line integral along the base (3.9) of the rectangle will follow once we show that the line integrals along the sides vanish,

$$\lim_{t\to\infty} \int_{\pm t+iy}^{\pm t+it} (cu)^{-k} e\left(\frac{-m}{c^2u} - nu\right) du = 0, \tag{3.10}$$

as well as across the top,

$$\lim_{t\to\infty}\int_{-t+it}^{t+it}(cu)^{-k}e\left(\frac{-m}{c^2u}-nu\right)du=0. \tag{3.11}$$

In both integrals, because $u\in\mathbb{H}$, we know $-m/c^2u\in\mathbb{H}$ since this is a positive multiple of a Möbius transformation; likewise, since $n\le 0$, either $-nu\in\mathbb{H}$ or $-nu=0$. In both cases, the exponential factor is bounded by 1. Furthermore, the polynomial factor $|cu|^{-k}$ is bounded by t^{-k} along both contours, so both integrals are bounded by $2t^{-k+1}$, which suffices to show (3.10) and (3.11). By (3.8), this shows that the n'th Fourier coefficient of $P_m(z)$ is precisely

$$p_m(n)=\delta(m,n)+\sum_{\substack{c>0\\ q|c}}S_{\overline{\chi}}(m,n;c)\int_{-\infty+iy}^{\infty+iy}(cu)^{-k}\exp\left(\frac{-2\pi im}{c^2u}-2\pi inu\right)du. \tag{3.12}$$

Now, it remains to recognize the Bessel function in the integral. In Appendix A, we verify that for any $\delta>0$, the following is a suitable integral representation of the J_ν-Bessel function:

$$J_\nu(z)=\frac{\left(\frac{1}{2}z\right)^\nu}{2\pi i}\int_{\delta+i\infty}^{\delta-i\infty}t^{-\nu-1}\exp\left(t-\frac{z^2}{4t}\right)dt. \tag{3.13}$$

Accordingly, we transform the integral in (3.12) via $t=-2\pi inu$ and $du=-dt/2\pi in$, which yields

$$\frac{(2\pi in)^{k-1}}{c^k}\int_{2\pi ny-i\infty}^{2\pi ny+i\infty}t^{-k}\exp\left(t-\frac{4\pi^2mn}{c^2t}\right)dt.$$

This has the shape of (3.13) for $v=k-1$ and $z=4\pi\sqrt{mn}/c$, which means the integral in (3.12) is precisely

$$\frac{(2\pi in)^{k-1}}{c^k}\frac{2\pi i}{\left(\frac{1}{2}\cdot\frac{4\pi}{c}\sqrt{mn}\right)^{k-1}}J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)=\left(\frac{n}{m}\right)^{(k-1)/2}\frac{2\pi\cdot i^k}{c}J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

This yields (3.4). □

4. The Petersson inner product

In this section, we construct the Petersson inner product on the space of cusp forms. This inner product is an integral with respect to the hyperbolic measure, which we now describe.

Lemma 4.1. *On the upper-half plane $\mathbb{H}$, the hyperbolic measure $d\mu$, defined by*

$$d\mu(z):=\frac{dx\,dy}{y^2}, \tag{4.1}$$

is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$.

The differential form $d\mu(z)$ defined in (4.1) is a volume form on $\mathbb{H}$, which means the measure of a Borel set $A\subseteq\mathbb{H}$ is defined to be its integral with respect to this form, i.e.

$$\mu(A):=\int_A d\mu(z).$$

That $d\mu(z)$ is $\mathrm{SL}_2(\mathbb{Z})$ invariant means that $\mu(A)=\mu(\gamma A)$ for all $\gamma\in\mathrm{SL}_2(\mathbb{Z})$.

Proof of Lemma 4.1. Fix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and write $\gamma(z) = \gamma(x_2 + iy_2) = x_1 + iy_1$. For any Borel set $A \subseteq \mathbb{H}$, we compute

$$\mu(\gamma A) = \int_{\gamma A} \frac{dx_1 dy_1}{y_1^2} = \int_A \frac{\partial(x_1, y_1)}{\partial(x_2, y_2)} \frac{dx_2 dy_2}{y_1^2}, \tag{4.2}$$

using the change of variables formula. Using the Cauchy-Riemann equations, we can compute that the Jacobian of a holomorphic function $f(x+iy) = u(x,y) + iv(x,y)$ is precisely $|f'(z)|^2$,

$$f'(z)\overline{f'(z)} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \frac{\partial u}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial v}{\partial x}\left(-\frac{\partial u}{\partial y}\right) = \frac{\partial(u,v)}{\partial(x,y)}.$$

In our case, the Jacobian in (4.2) is therefore $|\gamma'(z)|^2$, which we compute as follows:

$$\gamma'(z) = \frac{d}{dz}\left(\frac{az+b}{cz+d}\right) = \frac{(cz+d)a - (az+b)c}{(cz+d)^2} = \frac{1}{(cz+d)^2}.$$

Next we compute that $y_1 = y_2/|cz+d|^2$, since

$$y_1 = \Im \frac{a(x_2+iy_2)+b}{c(x_2+iy_2)+d} = \Im \frac{(ax_2+b)+i(ay_2)}{(cx_2+d)+i(cy_2)} \cdot \frac{(cx_2+d)-i(cy_2)}{(cx_2+d)-i(cy_2)} = \frac{y_2}{|cz+d|^2}.$$

So we can continue computing:

$$\int_A \frac{\partial(x_1, y_1)}{\partial(x_2, y_2)} \frac{dx_2 dy_2}{y_1^2} = \int_A \frac{1}{|cz+d|^4} \frac{dx_2 dy_2}{y_2^2/|cz+d|^4} = \int_A \frac{dx_2 dy_2}{y_2^2} = \mu(A), \tag{4.3}$$

so $\mu(\gamma A) = \mu(A)$ as needed. □

We denote by $M_k(q,\chi)$ (resp. $S_k(q,\chi)$) the space of modular (resp. cusp) forms of weight k on $\Gamma_0(q)$, with character χ.

Proposition 4.2. *Define the Petersson inner product on* $M_k(q,\chi)$ *to be*

$$\langle f, g\rangle := \int_{\Gamma_0(q)\backslash\mathbb{H}} f(z)\overline{g(z)} y^k d\mu, \tag{4.4}$$

where the integral is taken over a fundamental domain for the action of $\Gamma_0(q)$ *on* $\mathbb{H}$*. Then:*

(1) $\langle f, g\rangle$ *is well-defined;*
(2) $\langle f, g\rangle$ *is finite, and converges absolutely, as soon as one of* f *or* g *is a cusp form;*
(3) $S_k(q,\chi)$ *is a finite-dimensional Hilbert space with the Petersson inner product.*

Accordingly, we define the Petersson norm $\|f\| := \sqrt{\langle f, f\rangle}$.

Proof. For any $\gamma \in \Gamma_0(q)$, we compute that

$$f(\gamma z)\overline{g(\gamma z)}(\Im\gamma z)^k = \chi(\gamma)(cz+d)^k f(z) \cdot \overline{\chi(\gamma)(cz+d)^k g(z)} \cdot \frac{(\Im z)^k}{|cz+d|^{2k}},$$

which is exactly $f(z)\overline{g(z)}(\Im z)^k$. This shows that the integrand in (4.4) in $\Gamma_0(q)$ invariant, and is therefore independent of the choice of fundamental domain for the action of $\Gamma_0(q)$ on $\mathbb{H}$. Likewise, we computed in Lemma 4.1 that $d\mu$ is $\Gamma_0(q)$ invariant, which implies that the entire integral is well-defined. Since cusp forms decay exponentially at the cusps, the integrand is small in a neighborhood of the (finitely many) cusps, and bounded in the compact region outside these neighborhoods. This implies that the inner product is finite so long as one argument is a cusp form. It's straightforward to check that $S_k(q,\chi)$ is a finite dimensional

Hilbert space with this inner product, so long as one knows various fundamental facts about modular forms (e.g. the space of cusp forms is finite dimensional.) □

5. Projection of cusp forms onto Poincaré series

We proved in Section 3 that the Poincare series $P_m(z)$ with $m \geq 1$ are cusp forms. Now we verify that the span of the Poincaré series actually generates the space of cusp forms.

Proposition 5.1. *Let $f \in S_k(q, \chi)$ be a cusp form with Fourier expansion*

$$f(z) = \sum_{n \geq 1} a_f(n) e(nz). \tag{5.1}$$

Then for any $m \geq 1$, we have

$$\langle f, P_m \rangle = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}} a_f(m). \tag{5.2}$$

In particular, the Poincaré series with $m \geq 1$ span $S_k(q, \chi)$.

Proof. By definition of the Petersson inner product, we have

$$\langle f, P_m \rangle = \int_{\Gamma_0(q)\backslash \mathbb{H}} f(z) \left(\sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(q)} \chi(\gamma) \overline{j_\gamma(z)}^{-k} \overline{e(m\gamma z)} \right) y^k \frac{dxdy}{y^2}.$$

By modularity, $f(z) = f(\gamma z) \overline{\chi}(\gamma) j_\gamma(z)^{-k}$, so moving $f(z)$ inside the sum yields

$$\langle f, P_m \rangle = \int_{\Gamma_0(q)\backslash \mathbb{H}} \left(\sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(q)} |j_\gamma(z)|^{-2k} f(\gamma z) \overline{e(m\gamma z)} \right) y^k \frac{dxdy}{y^2}.$$

Using the identity $\Im(\gamma z)^k = y^k \cdot |j_\gamma(z)|^{-2k}$ and interchanging the sum and the integral yields

$$\langle f, P_m \rangle = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(q)} \int_{\Gamma_0(q)\backslash \mathbb{H}} f(\gamma z) \overline{e(m\gamma z)} \Im(\gamma z)^k \frac{dxdy}{y^2}.$$

In each summand, we apply the variable transformation $\gamma z \to z$. Since the measure $d\mu$ is $\mathrm{SL}_2(\mathbb{Z})$ invariant, this yields

$$\langle f, P_m \rangle = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(q)} \int_{\gamma(\Gamma_0(q)\backslash \mathbb{H})} f(z) \overline{e(mz)} \Im(z)^k \frac{dxdy}{y^2}.$$

Next, we observe [4, pp. 318] that the union of the images

$$\bigcup_{\gamma \in \Gamma_\infty \backslash \Gamma_0(q)} \gamma(\Gamma_0(q)\backslash \mathbb{H})$$

is a fundamental domain for $\Gamma_\infty \backslash \mathbb{H}$. So in computing the integral above, we may assume this domain coincides with the strip $0 \leq x < 1, y > 0$. This gives

$$\langle f, P_m \rangle = \int_0^\infty \int_0^1 f(z) \overline{e(mz)} y^{k-2} dxdy.$$

Next, we substitute the Fourier expansion (5.1) for $f(z)$ and simplify the exponentials. This introduces the product

$$\overline{e(mz)} e(nz) = e^{2\pi(-i)m(x-iy)} e^{2\pi in(x+iy)} = e^{2\pi i(n-m)x} e^{-2\pi(n+m)y},$$

so we separate the double integral as follows:

$$\langle f, P_m \rangle = \sum_{n\geq 1} a_f(n) \int_0^\infty y^{k-2} e^{-2\pi(n+m)y} dy \int_0^1 e^{2\pi i(n-m)x} dx.$$

We recognize that the integral with respect to x is 1 if $n = m$ and 0 otherwise, hence

$$\langle f, P_m \rangle = a_f(m) \int_0^\infty y^{k-2} e^{-4\pi m y} dy.$$

Next we recognize that the remaining integral with respect to y is the Γ function $\Gamma(z) := \int_0^\infty x^{z-1} e^{-x} dx$. If we make the substitution $x = 4\pi m y$, we may conclude that

$$\langle f, P_m \rangle = a_f(m) \int_0^\infty \left(\frac{x}{4\pi m}\right)^{k-2} e^{-x} \left(\frac{dx}{4\pi m}\right) = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}} a_f(m),$$

which verifies (5.2)

This implies that if f is orthogonal to P_m, then $a_f(m) = 0$. Furthermore, if f is orthogonal to every P_m, then $a_f(m) = 0$ for all $m \geq 1$, which implies $f = 0$. This implies that the orthogonal compliment of $\text{Span}\{P_m : m \geq 1\}$ within $S_k(q, \chi)$ is 0. Since the space spanned by the Poincaré series is closed (this uses the finite-dimensionality of $S_k(q, \chi)$) it follows that $\text{Span}\{P_m : m \geq 1\} = S_k(q, \chi)$. □

6. Petersson's trace formula

We now present Petersson's trace formula.

Theorem 6.1. *Let $\mathscr{F}$ be any orthonormal basis of $S_k(q, \chi)$. For any $n, m \geq 1$, we have*

$$\frac{\Gamma(k-1)}{(4\pi\sqrt{mn})^{k-1}} \sum_{f\in\mathscr{F}} a_f(n)\overline{a_f(m)} = \delta(m,n) + 2\pi i^k \sum_{\substack{c>0 \\ q|c}} c^{-1} S_{\overline{\chi}}(m,n;c) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right). \tag{6.1}$$

Proof. Since P_m is in $S_k(q, \chi)$, we can expand it as a linear combination of the $f \in \mathscr{F}$, namely we have

$$P_m = \sum_{f\in\mathscr{F}} \langle P_m, f \rangle f. \tag{6.2}$$

Petersson's trace formula will follow from equating the n'th Fourier coefficients on both sides of (6.2) in two different ways: on the LHS, the n'th Fourier coefficient is given explicitly by (3.4) to be

$$\delta(m,n) + \left(\frac{n}{m}\right)^{\frac{k-1}{2}} 2\pi \cdot i^k \sum_{\substack{c>0 \\ q|c}} c^{-1} S_{\overline{\chi}}(m,n;c) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

And on the RHS, using the inner product formula (5.2), we can compute

$$\begin{aligned}\sum_{f\in\mathscr{F}} \langle P_m, f\rangle f &= \sum_{f\in\mathscr{F}} \overline{\langle f, P_m\rangle} f \\ &= \sum_{f\in\mathscr{F}} \overline{\frac{\Gamma(k-1)}{(4\pi m)^{k-1}} a_f(m)} \sum_{n\geq 1} a_f(n) e(nz) \\ &= \frac{\Gamma(k-1)}{(4\pi m)^{k-1}} \sum_{f\in\mathscr{F}} \sum_{n\geq 1} a_f(n)\overline{a_f(m)} e(nz),\end{aligned}$$

which has n'th Fourier coefficient equal to

$$\frac{\Gamma(k-1)}{(4\pi m)^{k-1}} \sum_{f\in\mathscr{F}} a_f(n)\overline{a_f(m)}.$$

Equating these expressions for the n'th Fourier coefficient yields

$$\frac{\Gamma(k-1)}{(4\pi m)^{k-1}} \sum_{f\in\mathscr{F}} a_f(n)\overline{a_f(m)} = \delta(m,n) + \left(\frac{n}{m}\right)^{\frac{k-1}{2}} 2\pi i^k \sum_{\substack{c>0\\ q|c}} c^{-1} S_{\overline{\chi}}(m,n;c) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

Multiplying this by $(m/n)^{(k-1)/2}$ yields Petersson's trace formula (6.1). □

7. Toy application of Petersson's formula: a convergent series for $\tau(n)$

In Theorem 6.1, take $S_k(q,\chi) = S_{12}(1,1)$, which is the space of cusp forms of weight 12 on $\mathrm{SL}_2(\mathbb{Z})$. This space is one-dimensional, and is spanned by the modular discriminant

$$\Delta(z) = e(z)\prod_{n\geq 1}(1-e(nz))^{24} =: \sum_{n\geq 1} \tau(n) e(nz). \tag{7.1}$$

An orthonormal basis for the space spanned by $\Delta(z)$ is given by $f(z) = \Delta(z)/\|\Delta\|$. So, applying Petersson's trace formula with $\mathscr{F} = \{f\}$ yields

$$\frac{\Gamma(11)}{(4\pi\sqrt{mn})^{11}} \cdot \frac{\tau(m)\tau(n)}{\|\Delta\|^2} = \delta(m,n) + 2\pi \sum_{c\geq 1} c^{-1} S_1(m,n;c) J_{11}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

Specializing this formula to $m=1$ and $n>1$ yields a convergent series for $\tau(n)$,

$$\tau(n) = \frac{2^{23}\pi^{12} n^{11/2}\|\Delta\|^2}{10!} \sum_{c\geq 1} c^{-1} S_1(1,n;c) J_{11}\left(\frac{4\pi\sqrt{n}}{c}\right).$$

Note that the Ramanujan conjecture implies $|\tau(n)| \ll n^{\frac{11}{2}+\epsilon}$, so this series is $O(n^\epsilon)$.

Appendix A. Useful integral representation of the J_ν-Bessel function

Schläfli's integral representation [6, Eq. 10.9.19] for the Bessel function of the first kind, denoted J_ν, states that

$$J_\nu(z) = \frac{\left(\frac{1}{2}z\right)^\nu}{2\pi i}\int_{-\infty}^{(0+)} t^{-\nu-1}\exp\left(t - \frac{z^2}{4t}\right) dt.$$

Equivalently, this can be written as the integral over the clockwise $\supset$-shaped contour from $-\infty + ih$, to $h + ih$, to $h - ih$, to $-\infty - ih$, where $h > 0$ is a fixed parameter. Denote this contour by $\supset_h$, so our starting point will be the integral

$$\mathcal{I}(h; z) := \int_{\supset_h} t^{-\nu-1} \exp\left(t - \frac{z^2}{4t}\right) dt. \tag{A.1}$$

Here we verify that this has the equivalent integral representation that we utilized in Proposition 3.2 to compute the Fourier coefficients of the Poincaré series P_m.[3] To simplify our calculations, we assume that $z, v \in \mathbb{R}_{>0}$, which are true for our application. A reader who seeks true enlightenment is encouraged to sketch the contours that arise in our argument.

Lemma A.1. *The integral $\mathcal{I}(h; z)$ can also be written as*

$$\mathcal{I}(h; z) = \int_{h+i\infty}^{h-i\infty} t^{-\nu-1} \exp\left(t - \frac{z^2}{4t}\right) dt.$$

Proof. We argue in two steps:

(1) We'll shift the horizontal ray $(-\infty + ih, h + ih]$ to the vertical ray $(h + i\infty, h + ih]$.
(2) We'll shift the horizontal ray $[h - ih, -\infty - ih)$ to the vertical ray $[h - ih, h - i\infty)$.

Both of these contour shifts will be justified by Cauchy's integral theorem, as well as an appropriate limiting process.

For step (1), let us take $T > h$. By Cauchy's integral theorem applied to the square with vertices

$$-T + ih, \qquad -T + iT, \qquad h + iT, \qquad h + ih,$$

it suffices to verify the following limits:

$$\lim_{T\to\infty} \int_{-T+ih}^{-T+iT} t^{-\nu-1} \exp(t) \exp\left(-\frac{z^2}{4t}\right) dt = 0, \tag{A.2}$$

as well as

$$\lim_{T\to\infty} \int_{-T+iT}^{h+iT} t^{-\nu-1} \exp(t) \exp\left(-\frac{z^2}{4t}\right) dt = 0. \tag{A.3}$$

For the integrands in (A.2), we have $|t|^{-\nu-1} \le T^{-\nu-1}$ and $|\exp(t)| = \exp(-T)$. If we write $t = a + ib$, then we can compute that

$$\Re\left(-\frac{z^2}{4t}\right) = \frac{-z^2 a}{4(a^2 + b^2)}, \tag{A.4}$$

which implies $|\exp(-z^2/4t)| \le \exp(z^2/4T)$. Combining these estimates, we can bound

$$\Big|\int_{-T+ih}^{-T+iT} t^{-\nu-1} \exp(t) \exp\left(-\frac{z^2}{4t}\right) dt\Big| \le T \cdot T^{-\nu-1} \cdot \exp(-T) \cdot \exp\left(\frac{z^2}{4T}\right) \to 0,$$

so (A.2) indeed holds. We now turn our attention to (A.3). Over this contour, we have $|t|^{-\nu-1} \le T^{-\nu-1}$ and $|\exp(t)| \le \exp(h)$, and by (A.4) we have $|\exp(-z^2/4t)| \le \exp(z^2/4T)$. So in this case, we can bound

$$\Big|\int_{-T+iT}^{h+iT} t^{-\nu-1} \exp(t) \exp\left(-\frac{z^2}{4t}\right) dt\Big| \le 2T \cdot T^{-\nu-1} \cdot \exp(h) \cdot \exp\left(\frac{z^2}{4T}\right) \to 0.$$

[3] This integral representation is well-known to the initiated, and is used throughout the literature, but I didn't see this proof in any of my standard references, so we include it here for completeness.

This finishes step (1).

We now turn two step (2). By Cauchy's integral theorem applied to the square with vertices

$$-T-iT, \qquad -T-ih, \qquad h-ih, \qquad h-iT,$$

it suffices to show that

$$\lim_{T\to\infty}\int_{-T-iT}^{-T-ih} t^{-\nu-1}\exp(t)\exp\left(-\frac{z^2}{4t}\right)dt = 0, \tag{A.5}$$

as well as

$$\lim_{T\to\infty}\int_{-T-iT}^{h-iT} t^{-\nu-1}\exp(t)\exp\left(-\frac{z^2}{4t}\right)dt = 0. \tag{A.6}$$

For the integral in (A.5), we have $|t|^{-\nu-1}\le T^{-\nu-1}$ and $|\exp(t)| = \exp(-T)$, and by (A.4) we have $|\exp(-z^2/4t)| \le \exp(z^2/8T)$. Combining these estimates yields (A.5). And in the integral in (A.6), we have $|t|^{-\nu-1}\le T^{-\nu-1}$ and $\exp(t)\le\exp(h)$, and by (A.4) we have $|\exp(-z^2/4t)|\le\exp(z^2/4T)$. Combining these estimates indeed yields (A.6), which finishes the proof. □

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